Bayesian model selection

for

Markov chains

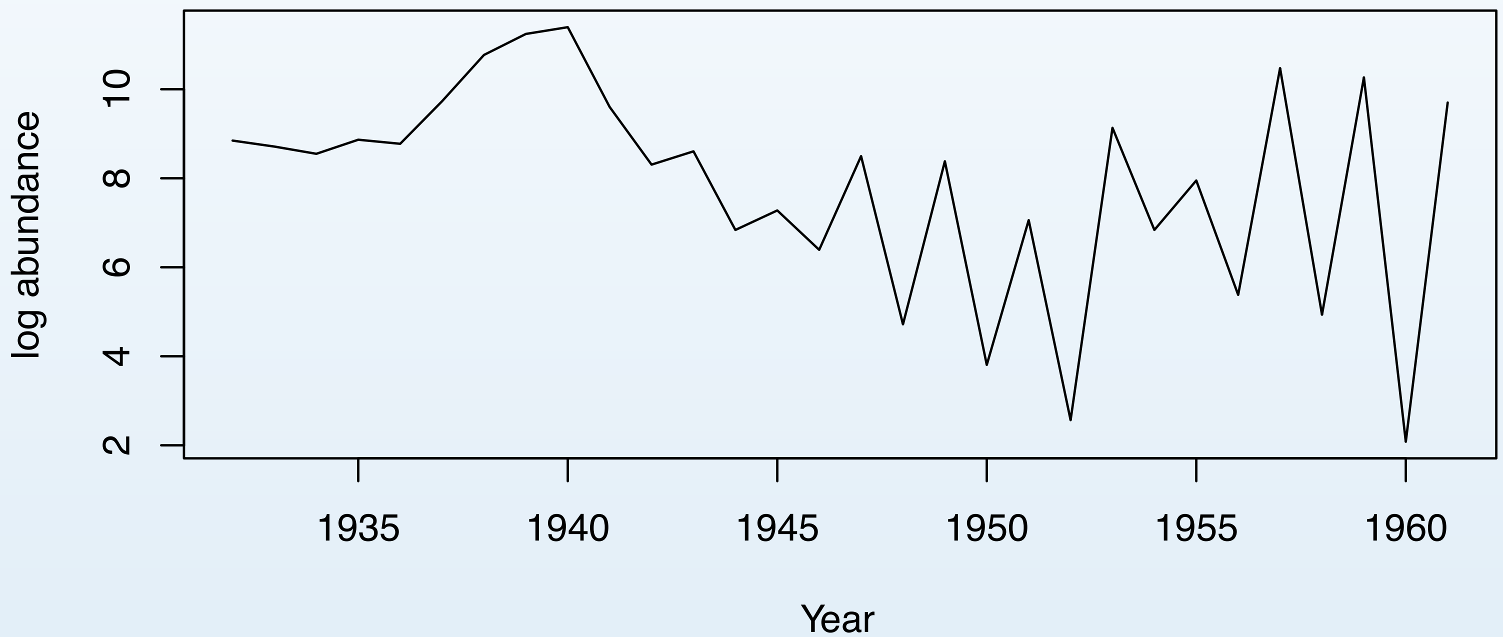
using

sparse probability vectors

Matthew Heiner[1], Athanasios Kottas[1], and Stephan Munch[2]

1. Department of Applied Mathematics and Statistics, University of California, Santa Cruz, California, USA
2. Fisheries Ecology Division, Southwest Fisheries Science Center, National Marine Fisheries Service, NOAA, Santa Cruz, California, USA

Pink Salmon time series



Data: Alaska Fisheries Science Center (2018), "AFSC/ABL: Pink salmon data collected at Sashin Creek Weir 1934-2002," NOAA National Centers for Environmental Information, https://inport.nmfs.noaa.gov/ inport/item/17256.

Image: U.S. Fish and Wildlife Service

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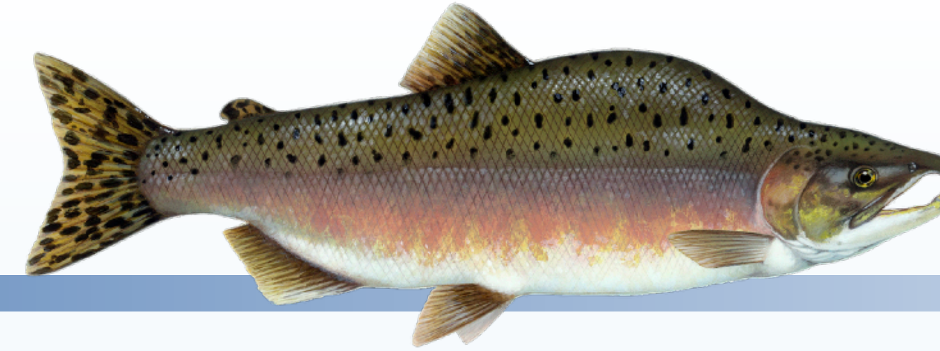

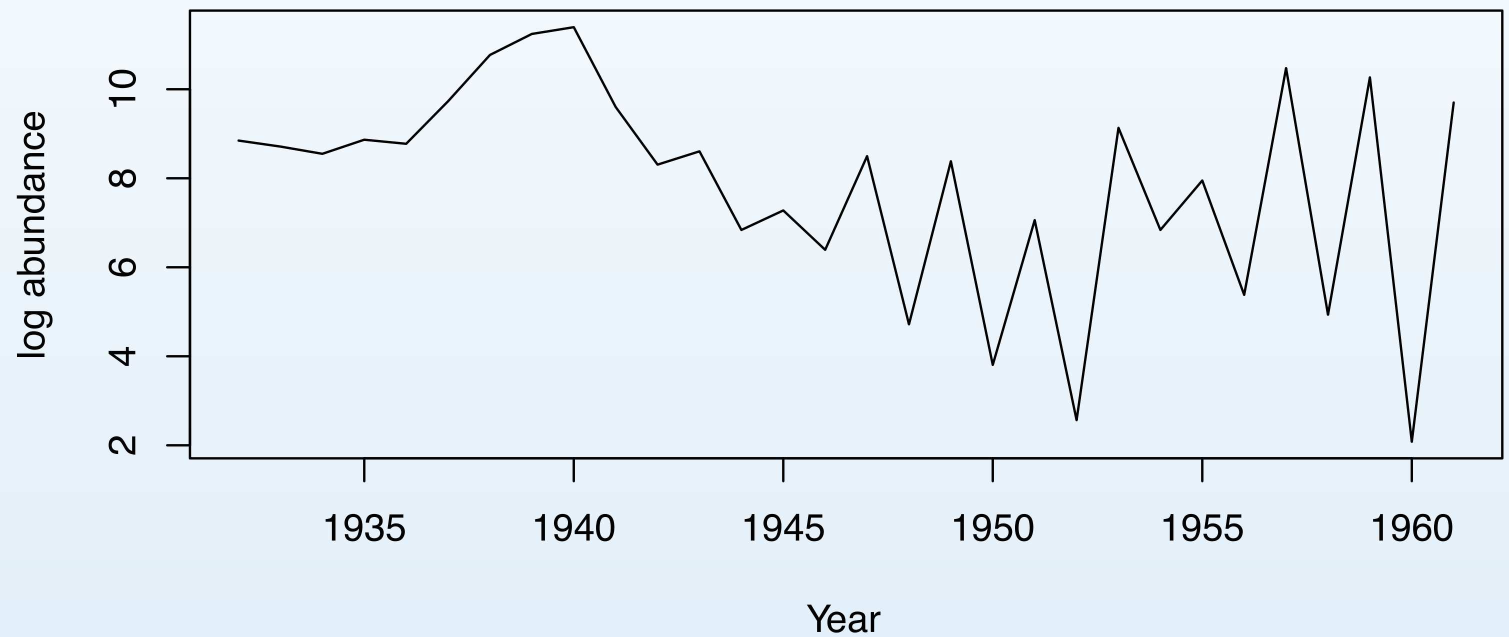

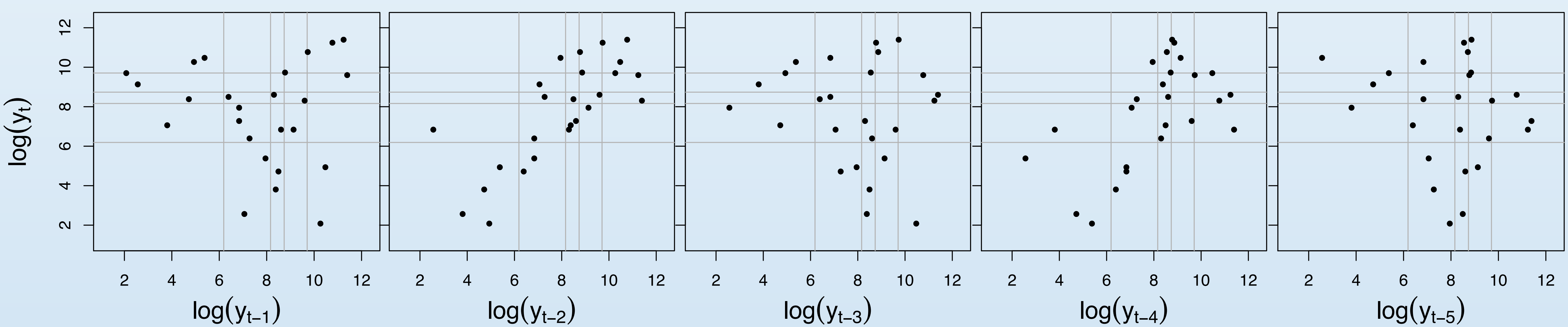

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Example with two states and three lags:

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$$Q = \begin{pmatrix} q_{1,1} & q_{1,2} \\ q_{2,1} & q_{2,2} \end{pmatrix}$$

first order
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Sparse probability vectors

Sparse Dirichlet mixture (SDM) prior

$$p(\boldsymbol{\theta}) \propto \mathrm{Dir}(\boldsymbol{\theta}; \boldsymbol{\alpha}) \times \sum_{k=1}^{K} \theta_k^{\beta}, \quad \beta > 1$$

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$$p(\boldsymbol{\theta} \mid \boldsymbol{n}) \quad \propto \quad \prod_{k=1}^{K} \theta_k^{n_k} \times \mathrm{Dir}(\boldsymbol{\theta}; \boldsymbol{\alpha}) \times \sum_{k=1}^{K} \theta_k^{\beta} \quad \propto \quad \mathrm{Dir}(\boldsymbol{\theta}; \boldsymbol{\alpha} + \boldsymbol{n}) \times \sum_{k=1}^{K} \theta_k^{\beta}$$

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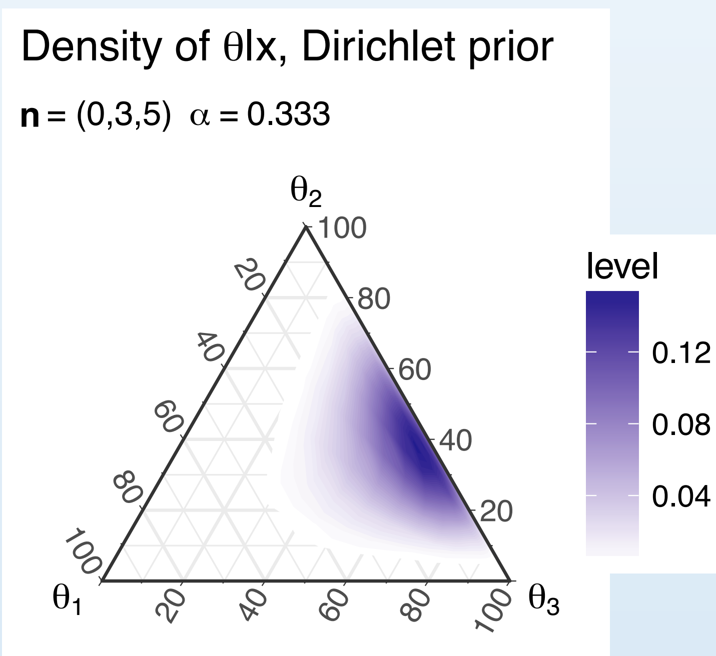
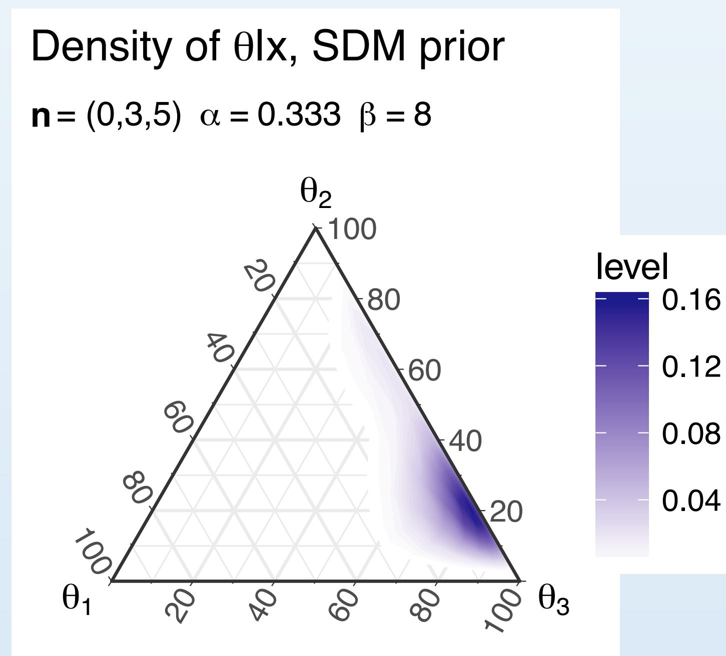
Smaller probabilities

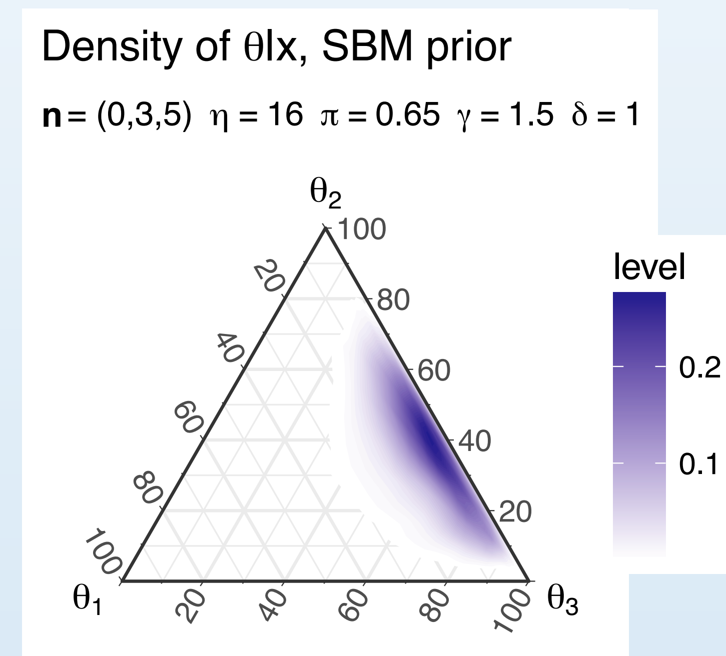
Larger probabilities

Sparse probability vectors

Multinomial data example







MTD extension

Let $J<R$ be a positive integer representing the highest order of mixing transition tensor with maximum lag R.

$$\Pr(s_t = i_0 \mid s_{t-1} = i_1, s_{t-2} = i_2, \dots, s_{t-R} = i_R, \boldsymbol{\Lambda}, \{\boldsymbol{\lambda}^{(j)}\}_{j=1}^{J}, \{\boldsymbol{Q}^{(j)}\}_{j=1}^{J}) = (\boldsymbol{\Omega})_{i_R, i_{R-1}, \dots, i_2, i_1, i_0}$$

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$$+\Lambda_J \sum_{\ell_1<\ldots<\ell_J} \ldots \sum \lambda_{(\ell_1,\ldots,\ell_J)}^{(J)} Q^{(J)}(s_t = i_0 \mid s_{t-\ell_1} = i_{\ell_1}, \ldots, s_{t-\ell_J} = i_{\ell_J})$$

MTD extension

Free parameter count:

K	R	J	Λ	λ	Q	total	unrestricted
5	3	2	1	4	120	125	500
5	4	3	2	11	620	633	2,500
2	7	4	3	94	30	127	128
3	5	3	2	22	78	102	486
7	5	3	2	22	2,394	2,418	100,842
7	5	2	1	13	336	350	100,842

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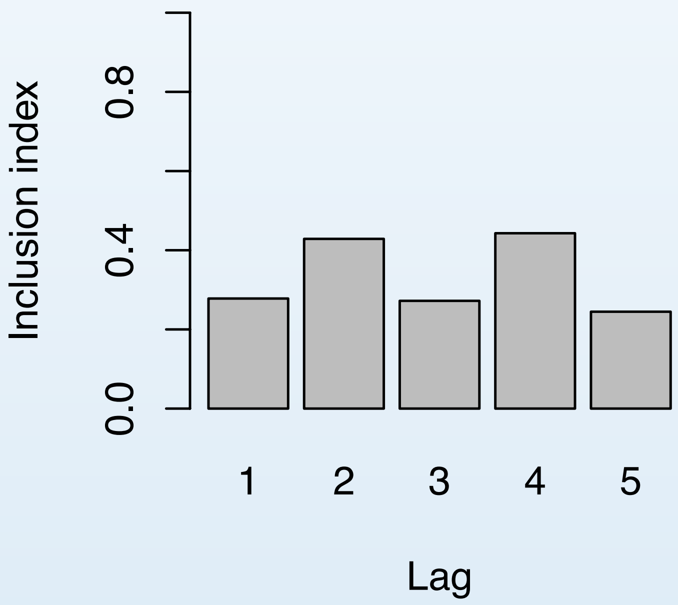
Lag inclusion inference

Dirichlet on mixing parameters



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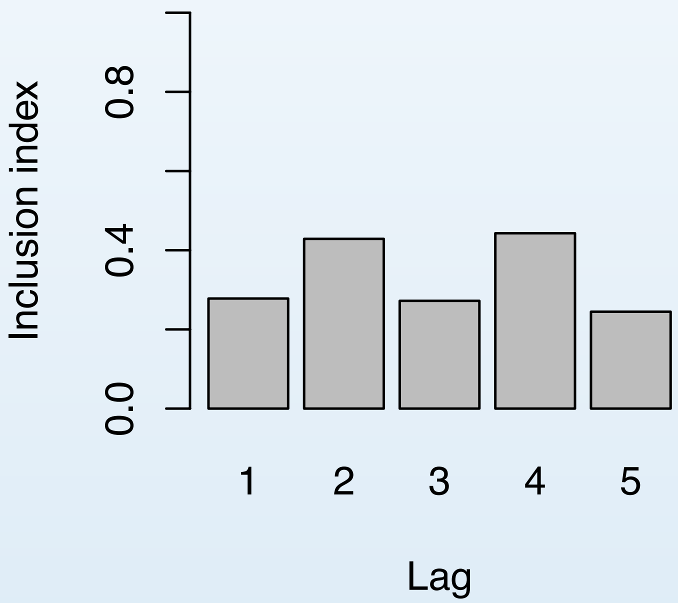
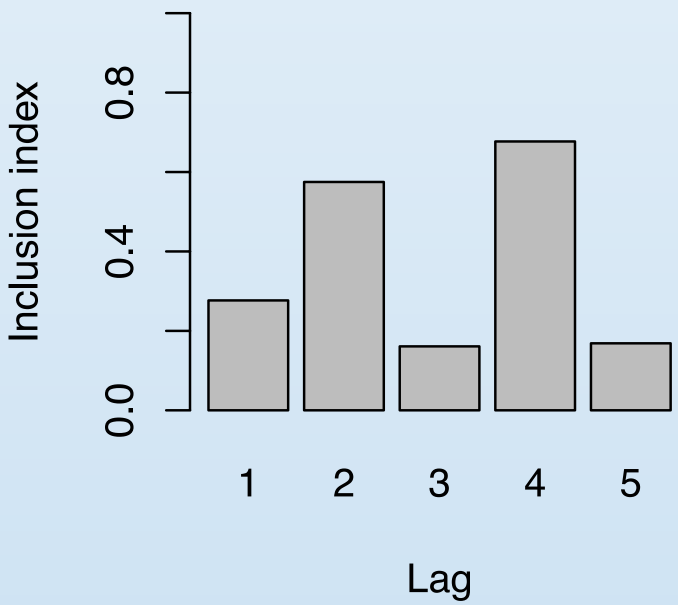


SDM on mixing parameters



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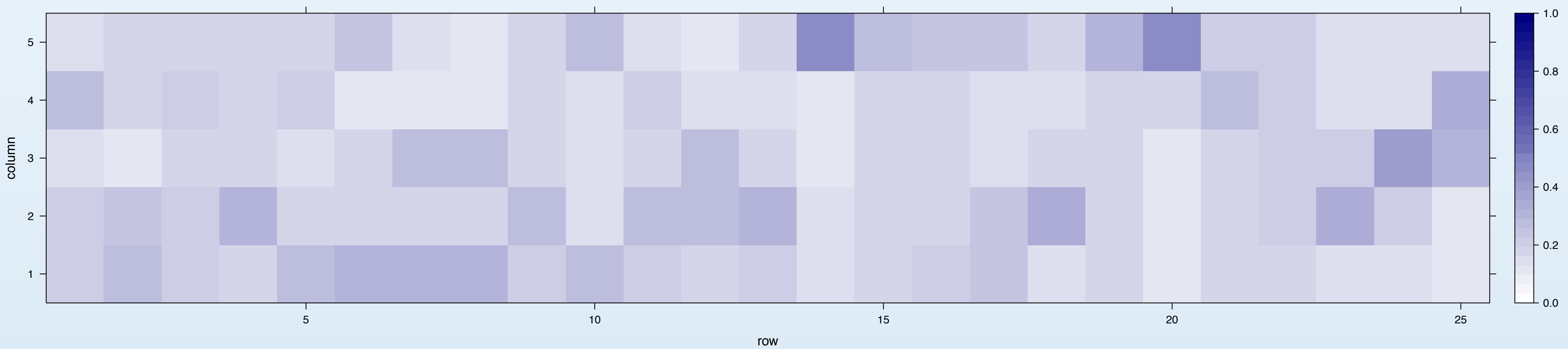
Posterior mean of $\boldsymbol{Q}$:

Second order with lags 2 and 4 most used

Dirichlet on mixing parameters (.67 for order 2)



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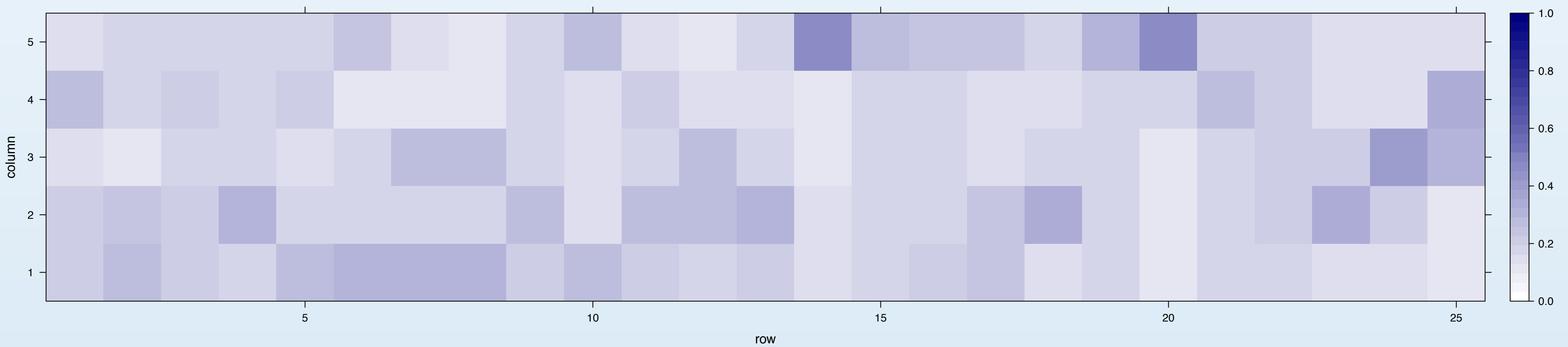
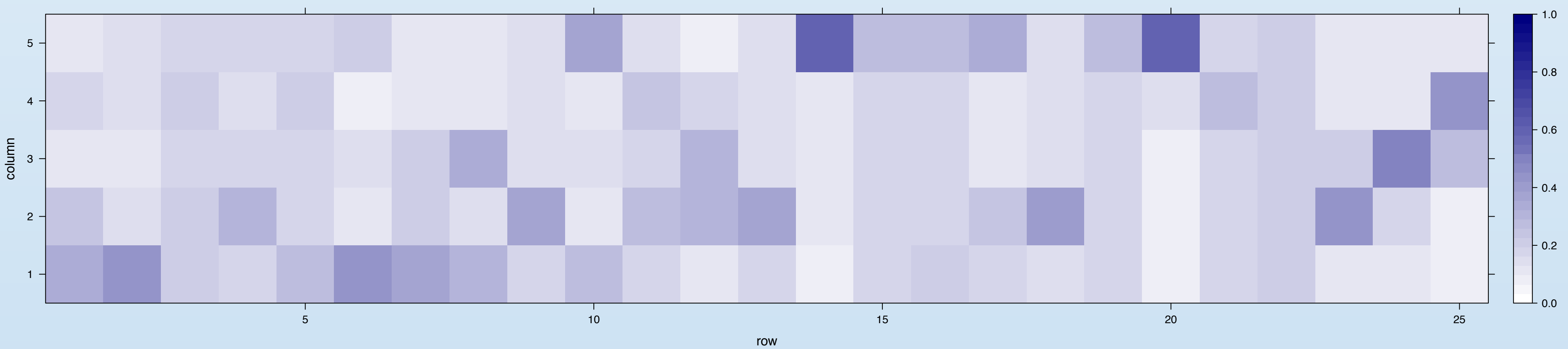


SDM on mixing parameters (.86 for order 2)



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Connor, Robert J., and James E. Mosimann. "Concepts of independence for proportions with a generalization of the Dirichlet distribution." *Journal of the American Statistical Association* 64.325 (1969): 194-206.

Raftery, Adrian E. "A model for high-order Markov chains." *Journal of the Royal Statistical Society. Series B (Methodological)* (1985): 528-539.

MTD extension

Hierarchical representation

$$(\boldsymbol{Q}^{(j)})_{i_j,i_{j-1},\ldots,i_1,\cdot} \sim \mathrm{Dir}(\boldsymbol{\alpha}_{\boldsymbol{Q}^{(j)}}) \quad \forall \quad (i_j, i_{j-1}, \ldots, i_1) \in \{1, \ldots, K\}^j \,,$$

$$\boldsymbol{\Lambda} \sim \mathrm{Dir}(\boldsymbol{\alpha}_\Lambda)\,, \qquad \boldsymbol{\lambda}^{(j)} \sim \mathrm{Dir}(\boldsymbol{\alpha}_{\lambda^{(j)}})\,,$$

$$\Pr(Z_t = j \mid \boldsymbol{\Lambda}) = \Lambda_j \,, \qquad \Pr(\boldsymbol{z}_t = (\ell_1, \ldots, \ell_j) \mid Z_t = j, \boldsymbol{\lambda}^{(j)}) = \lambda^{(j)}_{(\ell_1,\ldots,\ell_j)} \,,$$

$$\Pr(s_t = i_0 \,| s_{t-1} = i_1, s_{t-2} = i_2, \ldots, s_{t-R} = i_R, Z_t = j, z_t^{(j)} = (\ell_1, \ldots, \ell_j), \boldsymbol{Q}^{(j)}) = (\boldsymbol{Q}^{(j)})_{i_{\ell_j}, i_{\ell_{j-1}}, \ldots, i_{\ell_1}, i_0}$$